Wiener Index of Generalized Prism Garphs

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Abstract— In a connected graph G where V is the vertex set of G and E is the edge set of G, and the distance between two vertices of G is the total number of edges in the minimum path between them. The Wiener index is represented by $W(G)$ is the sum of all pairwise distances of vertices of the graph G, that is $W(G) = \frac{1}{2}\sum_{u,v\epsilon V(G)} d(u,v)$. In this paper the Wiener index for the families of generalized prism graphs $Z_{n,1}, Z_{n,2}$ and $Z_{n,3}$ are determined.

Index Terms— Connected Graph, Wiener index, Generalized Prism graphs $Z_{n,1}, Z_{n,2}$ and $Z_{n,3}$.

1 Introduction

THE distance between two vertices u and v in a connected graph G is equal to the number of edges in the minimum path between them. Since a long time ago this concept has been known and recently a lot of development has been achieved in this subject. Distance is related to many topological indices such as Wiener Index. Wiener index relates physio-chemical properties of organic materials and it is also studied as a subject by many mathematicians and chemists [3]. Initially the Wiener Index was applied to predict the boiling points of Alkanes [7]. Wiener called it the path number. Let $G(V, E)$ here V denotes vertex set and E indicates edges set of G. The Wiener index of G was first introduced by American chemist Harold Wiener in 1947 [15] in the study of relations between the structure organic compounds and their properties. Wiener index is denoted it by $W(G)$ i.e, $W(G) = \frac{1}{2}\sum_{u,v\epsilon V(G)} d(u,v)$ where u, v are vertices of G, is the sum of all pairwise distances of G [3]. The average distance is commonly used in specific fields of science such as physics, astronomy, engineering, social sciences and some other fields. More details on graph theory and its chemical applications can be found [7]. From the past thirty years mathematical relationships and chemical applications of the Wiener index have been highly concentrated [5]. Wiener specified Wiener index only for Alkanes. Mathematical work on Wiener Index is started in 1970's [6]. In 1971, Hosoya was the first person who introduced the relationship between Wiener Index and the distances in molecular graph. He also observed that the Wiener Index is equal to the half of the sum of all elements of the distance matrix of the related molecular graph. After this the Wiener index of cyclic molecules can be determined [9].

In 1975, Rouvray calculated the sum of the elements of the distance matrix, understanding it as a new topological index.
In the 1970′s and 1980′s many studies of the Wiener index were under taken [1], [7]. After its successful introduction and implemention, some more topological indices of chemical graphs, based on data in the distance matrix of the graph, have been introduced later to Wiener's work. The most common topological index is Wiener Index, now a days 200 topological indices has been used in chemistry [10]. For further studies and results we refer [4], [8], [12], [13] and [14]. The aim of this paper is to calculate Wiener index for the graph of generalized prism graphs $Z_{n,1}, Z_{n,2}$ and $Z_{n,3}$.

2 Generalized Prism Graphs

Definition 2.1. The generalized prism graph $Z_{n,s}, s \geq 1, n \geq s$ have vertex set $\{(i,j): i = 1,2 \text{ and } j = 1, 2, \cdots, n\}$ and an edge set $\{((i,j),(i,j\pm 1))\} \cup \{((1,i),(2,i+\delta)) \big| \delta: \delta = -\left[\frac{s-i}{2}\right], \cdots 0, \cdots [\frac{s}{2}]\}$.
See [2], [11] for more details.
So, the coming sections are about the computations of Wiener index of generalized prism graphs $Z_{8,1}, Z_{8,2}$ and $Z_{8,3}$. Following figures are of generalized prism graphs $Z_{8,1}, Z_{8,2}$ and $Z_{8,3}$.

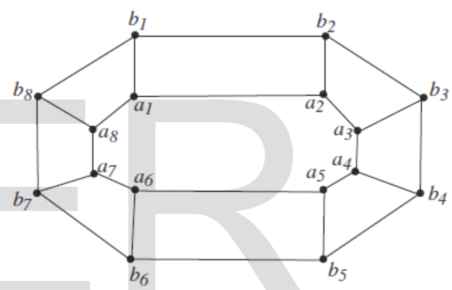

Figure: 2.1. The graph of $Z_{8,1}$

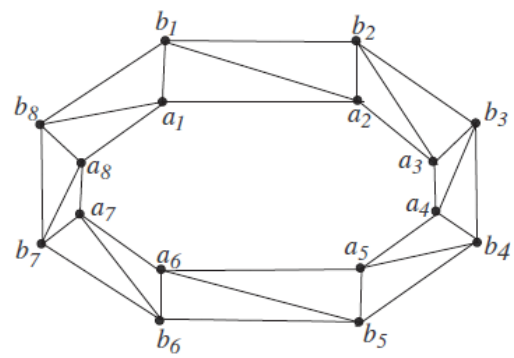

Figure: 2.2. The graph of $Z_{8,2}$

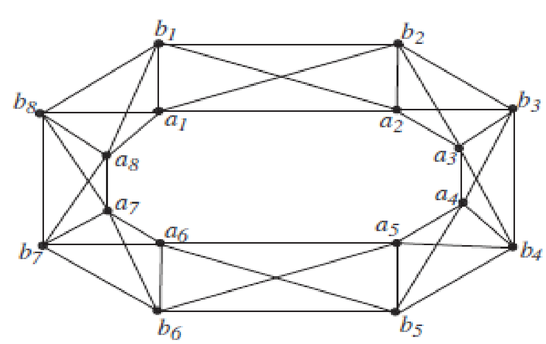

Figure: 2.3. The graph of $Z_{8,3}$

2.1 Wiener Index of $Z_{n,1}$

Theorem 2.1.1. The Wiener index of generalized prism graphs $Z_{n,1}$ for even values is $\frac{n^2}{2}(n+2)$.

Proof:

Firstly, we select the vertex b_1 on the outer cycle and determine its distance from each of the vertex on the outer and inner cycles respectively. Firstly, the distance from b_1 to each outer cycle vertex is:

i. For $2 \le j \le p+1$
$b_1 \to b_2 \to b_3 \to \cdots \to b_p \to b_j$ is a path of length $j-1$, hence $d(b_1, b_j) = j-1$.

ii. For $p+2 \le j \le 2p$, there is a path
$b_1 \to b_{2p} \to \cdots \to b_{j+1} \to b_j$ of length $2p-j+1$, hence $d(b_1, b_j) = 2p-j+1$.

So we can write,

$$d(b_1, b_j) = \begin{cases} j-1, & 2 \le j \le p+1 \\ 2p-j+1, & p+2 \le j \le 2p \end{cases}$$

Now, the distances from b_1 to each inner cycle vertex are:

i. For $1 \le j \le p+1$
$b_1 \to a_1 \to a_2 \to \cdots \to a_j$ is a path of length j,
hence $d(b_1, a_j) = j$.

ii. For $p+2 \le j \le 2p$, there is path
$b_1 \to a_1 \to a_{2p} \to \cdots \to a_{j+1} \to a_j$ of length $2p-j+2$, hence $d(b_1, a_j) = 2p-j+2$.

So we can write,

$$d(b_1, a_j) = \begin{cases} j, & 1 \le j \le p+1 \\ 2p-j+2, & p+2 \le j \le 2p \end{cases}$$

The distance of b_1 to each vertex of $Z_{n,1}$ is
$1+2+3+\cdots+(p-1)+p+(p-1)+(p-2)+\cdots+2+1$
And,
$1+2+3+\cdots+p+(p+1)+p+(p-1)+\cdots+4+3+2$
By adding above two equations we get,

$$3(1)+4(2)+4(3)+3(4)+\cdots+3p+(p+1) \quad \to (1)$$

Similarly, we select a vertex a_1 on the inner cycle and determine its distance from each of the vertex on the outer and inner cycles respectively, as given below.

$$d(a_1, b_j) = \begin{cases} j, & 1 \le j \le p+1 \\ 2p-j+2, & p+2 \le j \le 2p \end{cases}$$

$$d(a_1, a_j) = \begin{cases} j-1, & 2 \le j \le p+1 \\ 2p-j+1, & p+2 \le j \le 2p \end{cases}$$

The distance of a_1 to each vertex of $Z_{n,1}$ is
$1+2+3+\cdots+p+(p+1)+p+(p-1)+\cdots+4+3+2$
and
$1+2+3+\cdots+(p-1)+p+(p-1)+(p-2)+\cdots+2+1$
By adding above two equations we get,

$$3(1)+4(2)+4(3)+\cdots+4(p-1)+3p+(p+1) \quad \to (2)$$

We can apply the same process to each of vertex $Z_{n,1}$, due to symmetry there are n values, by using (1) and (2)

$$\sum_{u,v \in V(Z_{n,1})} d(u,v) = n\big[3(1)+4\big(2+3+\cdots+(p-1)\big)+3p+1(p+1)\big]+n[3(1)+4(2+3+\cdots+(p-1))+3p+1(p+1)]$$
$$= 4n[p^2+p]$$
$$= 4n[\left(\tfrac{n}{2}\right)^2+\tfrac{n}{2}] \quad \text{as } p=\tfrac{n}{2}$$
$$= n^2(n+2)$$

Thus the Wiener index of generalized prism graph is $W(Z_{n,1}) = \frac{n^2}{2}(n+2)$.

Theorem 2.1.2. The Wiener index of generalized prism graphs $Z_{n,1}$ for odd values is $\frac{n}{2}(n^2+2n-1)$.

Proof:

Firstly, we select the vertex b_1 on the outer cycle and determine its distance from each of the vertex on the outer and inner cycles respectively. Then the distance from b_1 to each outer cycle vertex is:

i. For $2 \le j \le p+1$
$b_1 \to b_2 \to b_3 \to \cdots \to b_p \to b_j$ is a path of length $j-1$, hence $d(b_1, b_j) = j-1$.

ii. For $p+2 \le j \le 2p+1$
$b_1 \to b_{2p+1} \to \cdots \to b_{j+1} \to b_j$ there is path of length $2p-j+2$, hence $d(b_1, b_j) = 2p-j+2$.

So we can write,

$$d(b_1, b_j) = \begin{cases} j-1, & 2 \le j \le p+1 \\ 2p-j+2, & p+2 \le j \le 2p+1 \end{cases}$$

Now, the distances from b_1 to each inner cycle vertex is:

i. For $1 \le j \le p+1$
$b_1 \to a_1 \to a_2 \to \cdots \to a_j$ is a path of length j,
hence $d(b_1, a_j) = j$.

ii. For $p+2 \le j \le 2p+1$, there is path
$b_1 \to a_1 \to a_{2p+1} \to \cdots \to a_{j+1} \to a_j$ of length $2p-j+3$,
hence $d(b_1, a_j) = 2p-j+3$.

So we can write,

$$d(b_1, a_j) = \begin{cases} j, & 1 \le j \le p+1 \\ 2p-j+3, & p+2 \le j \le 2p+1 \end{cases}$$

Thus the distance of b_1 to each vertex of $Z_{n,1}$ is
$1+2+3+\cdots+(p-1)+p+p+(p-1)+\cdots+3+2+1$
And,
$1+2+3+\cdots+p+(p+1)+(p+1)+p+\cdots+4+3+2$
By adding above two equations we get,

$$3(1)+4(2)+4(3)+\cdots+4(p-1)+4p+2(p+1) \quad \to (1)$$

Similarly, we select a vertex a_1 on the inner cycle and determine its distance from each of the vertex on the outer and inner cycles respectively, as given below.

$$d(a_1, b_j) = \begin{cases} j, & 1 \le j \le p+1 \\ 2p-j+3, & p+2 \le j \le 2p+1 \end{cases}$$

$$d(a_1, a_j) = \begin{cases} j-1, & 1 \le j \le p+1 \\ 2p-j+2, & p+2 \le j \le 2p+1 \end{cases}$$

The distance of a_1 to each vertex of $Z_{n,1}$ is
$1+2+3+\cdots+(p-1)+p+p+(p-1)+\cdots+3+2+1$
and
$1+2+3+\cdots+(p-1)+p+(p+1)+(p+1)+p+(p-1)+4+3+2$
By adding above two equations we get,

$$3(1)+4(2)+4(3)+\cdots+4(p-1)+4p+2(p+1) \quad \to (2)$$

We can apply the same process to each of vertex $Z_{n,1}$, due to symmetry there are n values, by using (1) and (2)

$$\sum_{u,v \in V(Z_{n,1})} d(u,v) = n[3(1)+4(2+3+4+\cdots+(p-1)+p)+2(p+1)]+n[3(1)+4(2+3+4+\cdots+(p-1)+p)+2(p+1)] = 2n[2p^2+4p+1]$$

$= 2n[\frac{2(n-1)^2}{2} + \frac{4(n-1)}{2} + 1]$ as $p = \frac{n-1}{2}$

$= 2n[\frac{n^2+2n-1}{2}]$

Thus the Wiener index of generalized prism graph is
$W(Z_n, 1) = \frac{n}{2}[\frac{n^2+2n-1}{2}]$

Illustration 2.1.3.
Figure 2.1 is the figure of generalized prism graph $Z_{n,1}$ where $n = 8$. The Wiener index for this graph is determined as follows.
We select a vertex b_1 on the outer cycle and determine its distance from each of the outer and inner cycles respectively, which are as follows
0,1,2,3,4,3,2,1
1,2,3,4,5,4,3,2
Now we select a vertex a_1 on the inner cycle and determine its distance from each of the vertices on the outer and inner cycles respectively which are
0,1,2,3,4,3,2,1
1,2,3,4,5,4,3,2
As the graph is symmetric in the similar way we can find the distance of each vertex on the outer and inner cycle from each vertex of graph is $8(40) = 320$ in each case $320 + 320 = 640$

$$W(Z_{n,1}) = 320$$

The sum of distances of each of the vertices on the graph is by substituting $n = 8$ in Theorem 2.2 is

$W(Z_{n,1}) = \frac{n^2}{2}(n+2)$ implies that $W(Z_{8,1}) = \frac{8^2}{2}(8+2) = 320.$

2.2 Wiener Index of $Z_{n,2}$

Theorem 2.2.1. The Wiener index of generalized prism graphs $Z_{n,2}$ for all values of n is $\frac{n^2}{2}(n+1)$.
Proof:
Case1: When n is even
Firstly, we select the vertex b_1 on the outer cycle and determine its distance from each of the vertex on the outer and inner cycles respectively. Then the distances from b_1 to each outer cycle vertex are:

i. For $2 \le j \le p+1$ there is path $b_1 \to b_2 \to b_3 \to \cdots \to b_p \to b_j$ of length $j-1$, hence $d(b_1, b_j) = j-1$.
ii. For $p+2 \le j \le 2p$ there is path of $b_1 \to b_{2p} \to \cdots \to b_{j+1} \to b_j$ of length $2p-j+1$, hence $d(b_1, b_j) = 2p-j+1$.

So we can write,

$$d(b_1, b_j) = \begin{cases} j-1, & 2 \le j \le p+1 \\ 2p-j+1, & p+2 \le j \le 2p \end{cases}$$

Now, the distances from b_1 to each inner cycle vertex is:

i. For $j = 1$, $d(b_1, a_j) = 1$
ii. For $2 \le j \le p+1$
$b_1 \to a_2 \to \cdots \to a_p \to a_j$ is a path of length j, hence $d(b_1, a_j) = j-1$.
iii. For $p+2 \le j \le 2p$
$b_1 \to a_1 \to a_{2p} \to a_{2p-1} \to \cdots \to a_j$ is a path of length $2p-j+2$, hence $d(b_1, a_j) = 2p-j+2$.

So we can write, $d(b_1, a_j) = \begin{cases} 1, & j = 1 \\ j-1, & 2 \le j \le p+1 \\ 2p-j+2, & p+2 \le j \le 2p \end{cases}$

The distance of b_1 to each vertex of $Z_{n,2}$ is
$1 + 2 + 3 + 4 + \cdots + (p-1) + p + (p-1) + \cdots + 3 + 2 + 1$
And,
$1 + 1 + 2 + 3 + \cdots + (p-1) + p + p + (p-1) + \cdots + 4 + 3 + 2$
By adding above two equations we get,
$4(1) + 4(2) + 4(3) + 4(4) + \cdots + 4(p-1) + 3p \quad \to (1)$
Lastly, we select a vertex a_1 on the inner cycle and determine its distance from each of the vertex on the outer and inner cycles respectively, as given below.

i. $d(a_1, b_j) = \begin{cases} j, & 1 \le j \le p \\ 2p-j+1, & p+1 \le j \le 2p \end{cases}$

ii. $d(a_1, a_j) = \begin{cases} j-1, & 2 \le j \le p+1 \\ 2p-j+1, & p+2 \le j \le 2p \end{cases}$

The distance of a_1 to each vertex of $Z_{n,2}$ is
$1 + 2 + 3 + 4 + \cdots + (p-1) + p + p + (p-1) + \cdots + 3 + 2 + 1$
and
$1 + 2 + 3 + 4 + \cdots + (p-1) + p + (p-1) + \cdots + 3 + 2 + 1$
By adding above two equations we get,
$4(1) + 4(2) + 4(3) + 4(4) + \cdots + 4(p-1) + 3p \quad \to (2)$
We can apply the same process to each of vertex $Z_{n,2}$, due to symmetry there are n values, by using (1) and (2)

$$\sum_{u,v \in V(Z_{n,1})} d(u,v) = n[4(1+2+3+\cdots+(p-1)) + 3p] + n[4(1+2+3+\cdots+(p-1)) + 3p]$$
$$= 2n[2p^2 + p]$$
$$= 2n[2\left(\frac{n}{2}\right)^2 + \frac{n}{2}] \quad \text{as } p = \frac{n}{2}$$
$$= 2n(\frac{n^2+n}{2})$$

Thus the Wiener index of generalized prism graph $Z_{n,2}$ is
$W(Z_{n,2}) = n(\frac{n^2+1}{2})$
Case 2: When n is odd
Firstly, we select the vertex b_1 on the outer cycle and determine its distance from each of the vertex on the outer and inner cycles respectively. Then the distances from b_1 to each outer cycle vertex is:

i. For $2 \le j \le p+1$
$b_1 \to b_2 \to b_3 \to \cdots \to b_p \to b_j$ is a path of length $j-1$, hence $d(b_1, b_j) = j-1$.
ii. For $p+2 \le j \le 2p+1$
$b_1 \to b_{2p+1} \to \cdots \to b_j$ is a path of length $2p-j+2$, hence $d(b_1, b_j) = 2p-j+2$.

So we can write,

$$d(b_1, b_j) = \begin{cases} j-1, & 2 \le j \le p+1 \\ 2p-j+2, & p+2 \le j \le 2p+1 \end{cases}$$

Now, the distances from b_1 to each inner cycle vertex is:

i. For $j = 1$, $d(b_1, a_j) = 1$
ii. For $2 \le j \le p+1$
$b_1 \to a_2 \to \cdots \to a_p \to a_j$ is a path of length $j-1$, hence $d(b_1, a_j) = j-1$.
iii. For $p+2 \le j \le 2p+1$
$b_1 \to a_1 \to a_{2p+1} \to \cdots \to a_j$ is a path of length $2p-j+3$, hence $d(b_1, a_j) = 2p-j+3$.

So we can write,

$$d(b_1,a_j)=\begin{cases}1, & j=1\\ j-1, & 2\le j\le p+1\\ 2p-j+3, & p+2\le j\le 2p+1\end{cases}$$

Thus the distance of b_1 to each vertex of $Z_{n,2}$ is

$1+2+3+\cdots+p+p+\cdots+3+2+1$

And,

$1+1+2+3+\cdots+p+(p+1)+p+\cdots+4+3+2$

By adding above two equations we get,

$4(1)+4(2)+4(3)+4(4)+\cdots+4p+(p+1) \quad \rightarrow (1)$

Lastly, we select a vertex a_1 on the inner cycle and determine its distance from each of the vertex on the outer and inner cycles respectively, as given below.

$$d(a_1,b_j)=\begin{cases}j, & 1\le j\le p+1\\ 2p-j+2, & p+2\le j\le 2p+1\end{cases}$$

$$d(a_1,a_j)=\begin{cases}j-1, & 2\le j\le p+1\\ 2p-j+2, & p+2\le j\le 2p+1\end{cases}$$

The distance of a_1 to each vertex of $Z_{n,2}$ is

$1+2+3+\cdots+p+(p+1)+p+\cdots+3+2+1$

and

$1+2+3+\cdots+p+p+\cdots+3+2+1$

By adding above two equations we get,

$4(1)+4(2)+4(3)+4(4)+\cdots+4p+(p+1) \quad \rightarrow (2)$

We can apply the same process to each of vertex $Z_{n,2}$, due to symmetry there are n values, by using (1) and (2)

$$\begin{aligned}\sum_{u,v\in V(Z_{n,1})} d(u,v) &= n[4(1+2+3+4+\cdots+p)+(p+1)]\\ &+n[4(1+2+3+4+\cdots+p)+(p+1)]\\ &=2n[2p^2+3p+1]\\ &=2n[\tfrac{2(n-1)^2}{2}+\tfrac{3(n-1)}{2}+1] \quad \text{as } p=\tfrac{n-1}{2}\\ &=2n[\tfrac{2n^2+2n}{4}]\end{aligned}$$

Thus the Wiener index of generalized prism graph is

$$W(Z_{n,2})=n\left[\frac{2n^2+2n}{4}\right]=\frac{n^2(n+1)}{2}$$

2.3 WIENER INDEX OF $Z_{n,3}$

Theorem 2.1.1. The Wiener index of generalized prism graphs $Z_{n,3}$ for even values is $\frac{n}{2}(n^2+2)$.

Proof:

Firstly, we select the vertex b_1 on the outer cycle and determine its distance from each of the vertex on the outer and inner cycles respectively. Then the distance from b_1 to each outer cycle vertex is:

i. For $2\le j\le p+1$
$b_1\rightarrow b_2\rightarrow b_3\rightarrow\cdots\rightarrow b_p\rightarrow b_j$ is a path of length $j-1$, hence $d(b_1,b_j)=j-1$.

ii. For $p+2\le j\le 2p$
$b_1\rightarrow b_{2p}\rightarrow b_{2p-1}\rightarrow\cdots\rightarrow b_j$ is a path of length $2p-j+1$, hence $d(b_1,b_j)=2p-j+1$.

So we can write,

$$d(b_1,b_j)=\begin{cases}j-1, & 2\le j\le p+1\\ 2p-j+1, & p+2\le j\le 2p\end{cases}$$

Now, the distances from b_1 to each inner cycle vertex is:

i. For $j=1$, $d(b_1,a_j)=1$

ii. For $2\le j\le p+1$
$b_1\rightarrow a_2\rightarrow a_3\rightarrow\cdots\rightarrow a_p\rightarrow a_j$ is a path of length $j-1$, hence $d(b_1,a_j)=j-1$.

iii. For $p+2\le j\le 2p$
$b_1\rightarrow a_{2p}\rightarrow a_{2p-1}\rightarrow\cdots\rightarrow a_j$ is a path of length $2p-j+1$, hence $d(b_1,a_j)=2p-j+1$.

So we can write,

$$d(b_1,a_j)=\begin{cases}1, & j=1\\ j-1, & 2\le j\le p+1\\ 2p-j+1, & p+2\le j\le 2p\end{cases}$$

The distance of b_1 to each vertex of $Z_{n,3}$ is

$1+2+3+\cdots+(p-1)+p+(p-1)+(p-2)+\cdots+2+1$

And,

$1+1+2+3+\cdots+(p-1)+p+(p-1)+\cdots+4+3+2+1$

By adding above two equations we get,

$5(1)+4(2)+4(3)+\cdots+4(p-1)+2p \quad \rightarrow (1)$

Similarly, we select a vertex a_1 on the inner cycle and determine its distance from each of the vertex on the outer and inner cycles respectively, as given below.

$$d(a_1,b_j)=\begin{cases}1, & j=1\\ j-1, & 2\le j\le p+1\\ 2p-j+1, & p+2\le j\le 2p\end{cases}$$

$$d(a_1,a_j)=\begin{cases}j-1, & 2\le j\le p+1\\ 2p-j+1, & p+2\le j\le 2p\end{cases}$$

The distance of a_1 to each vertex of $Z_{n,3}$ is

$1+1+2+3+\cdots+(p-1)+p+(p-1)+\cdots+4+3+2+1$

and

$1+2+3+\cdots+(p-1)+p+(p-1)+(p-2)+\cdots+2+1$

By adding above two equations we get,

$5(1)+4(2)+4(3)+4(4)+\cdots+4(p-1)+2p \quad \rightarrow (2)$

We can apply the same process to each of vertex $Z_{n,3}$, due to symmetry there are n values, by using (1) and (2)

$$\begin{aligned}\sum_{u,v\in V(Z_{n,1})} d(u,v) &= n\big[5(1)+4\big(2+3+\cdots+(p-1)\big)+2p\big]+\\ &\quad n\big[5(1)+4\big(2+3+\cdots+(p-1)\big)+2p\big]\\ &=2n[2p^2+1]\\ &=2n[2\left(\tfrac{n}{2}\right)^2+1] \quad \text{as } p=\tfrac{n}{2}\\ &=2n(\tfrac{n^2+2}{2})\end{aligned}$$

Thus the Wiener index of $Z_{n,3}$ for even value of n is

$$W(Z_{n,3})=\frac{n}{2}(n^2+2)$$

Theorem 2.1.2. The Wiener index of generalized prism graphs $Z_{n,3}$ for odd values is $\frac{n}{2}(n^2+1)$.

Proof:

Firstly, we select the vertex b_1 on the outer cycle and determine its distance from each of the vertex on the outer and inner cycles respectively. Then the distance from b_1 to each outer cycle vertex is:

i. For $2\le j\le p+1$
$b_1\rightarrow b_2\rightarrow b_3\rightarrow\cdots\rightarrow b_p\rightarrow b_j$ is a path of length $j-1$, hence $d(b_1,b_j)=j-1$.

ii. For $p+2\le j\le 2p+1$
$b_1\rightarrow b_{2p+1}\rightarrow b_{2p}\rightarrow\cdots\rightarrow b_j$ is a path of length $2p-j+2$, hence $d(b_1,b_j)=2p-j+2$.

So we can write,

$$d(b_1,b_j)=\begin{cases}j-1, & 2\le j\le p+1\\ 2p-j+2, & p+2\le j\le 2p+1\end{cases}$$

Now, the distances from b_1 to each inner cycle vertex is:

i. For $j=1$, $d(b_1,a_j)=1$

ii. For $2 \leq j \leq p+1$
$b_1 \to a_2 \to a_3 \to \cdots \to a_p \to a_j$ is a path of length $j-1$, hence $d(b_1, a_j) = j-1$.

iii. For $p+2 \leq j \leq 2p+1$
$b_1 \to a_{2p+1} \to a_{2p} \to \cdots \to a_j$ is a path of length $2p-j+2$, hence $d(b_1, a_j) = 2p-j+2$.

So we can write,

$$d(b_1, a_j) = \begin{cases} 1, & j = 1 \\ j-1, & 2 \leq j \leq p+1 \\ 2p-j+2, & p+2 \leq j \leq 2p+1 \end{cases}$$

Thus the distance of b_1 to each vertex of $Z_{n,3}$ is
$1+2+3+\cdots+(p-1)+p+p+(p-1)+\cdots+3+2+1$
And,
$1+2+3+\cdots+(p-1)+p+p+(p-1)+\cdots+3+2+1$
By adding above two equations we get,
$5(1)+4(2)+4(3)+2(4)+\cdots+4(p-1)+4p \quad \to (1)$
Lastly, we select a vertex a_1 on the inner cycle and determine its distance from each of the vertex on the outer and inner cycles respectively, as given below.

$$d(a_1, b_j) = \begin{cases} 1, & j = 1 \\ j-1, & 2 \leq j \leq p+1 \\ 2p-j+2, & p+2 \leq j \leq 2p+1 \end{cases}$$

$$d(a_1, a_j) = \begin{cases} j-1, & 2 \leq j \leq p+1 \\ 2p-j+2, & p+2 \leq j \leq 2p+1 \end{cases}$$

The distance of a_1 to each vertex of $Z_{n,3}$ is
$1+1+2+3+\cdots+(p-1)+p+p+(p-1)+\cdots+3+2+1$
and
$1+2+3+\cdots+(p-1)+p+p+(p-1)+3+2+1$
By adding above two equations we get,
$5(1)+4(2)+4(3)+\cdots+4(p-1)+4p \quad \to (2)$
We can apply the same process to each of vertex $Z_{n,3}$, due to symmetry there are n values, by using (1) and (2)

$$\begin{aligned} \sum_{u,v \in V(Z_{n,1})} d(u,v) &= n\left[5(1)+4\left(2+3+\cdots+(p-1)\right)+2p\right] + n\left[5(1)+4\left(2+3+\cdots+(p-1)\right)+2p\right] \\ &= 2n[2p^2+2p+1] \\ &= 2n\left[\frac{2(n-1)^2}{4}+\frac{2(n-1)}{2}+1\right] \quad \text{as } p = \frac{n-1}{2} \\ &= 2n\left[\frac{n^2+1}{2}\right] = n(n^2+1) \end{aligned}$$

Thus the Wiener index of $Z_{n,3}$ for odd values of n is
$W(Z_{n,3}) = \frac{n}{2}(n^2+1)$

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